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# Line hypergraphs

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**Abstract**

In this paper, we introduce a new multivalued function $\mathscr{L}$ called the line hypergraph. The function $\mathscr{L}$ generalizes two classical concepts at once, namely, of the line graph and the dual hypergraph. In terms of this function, proofs of some known theorems on line graphs can be unified and their more general versions can be obtained. Three such theorems are considered here, namely, the Berge theorem describing all hypergraphs with a given line graph $G$ in terms of clique coverings of $G$ (Berge, 1973, p. 400), the Krausz global characterization of line graphs for simple graphs (Krausz, 1943) and the Whitney theorem on isomorphisms of line graphs (Whitney, 1932).

## 1. Introduction

A new multivalued function $\mathscr{L}$ called the line hypergraph is introduced, which generalizes simultaneously the classical concepts of the line graph and the dual hypergraph. The necessary terminology and the definition of the function 'line hypergraph' are given in Sections 2 and 3, respectively. All values of this function $\mathscr{L}(H)$ are called line hypergraphs of a hypergraph $H$. Moreover, Section 3 contains a technical lemma on isomorphisms. In particular, the lemma transfers the statement of Lovasz about trivial isomorphisms of line graphs [13, p. 506] onto line hypergraphs. Theorem 2 from Section 4 describes the inverse image $\mathscr{L}^{-1}(G)$ in terms of clique covering of a hypergraph $G$. This is the main result of the paper, all other results are based on Theorem 2. For a simple graph $G$, Theorem 2 yields Proposition 1 from [2, p. 400] together with a criterion of isomorphism of elements in $\mathscr{L}^{-1}(G)$.

A characterization of values of the functions 'line hypergraph' for hypergraphs with a prescribed property $P$ is given in Section 5. The characterization generalizes the Krausz theorem [11] and is formulated in terms of clique coverings as well as the Krausz theorem. So, we call this characterization and the corresponding coverings the

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Krausz type characterization and Krausz coverings. In this section we introduce an important concept of the $P$-large clique for some given properties $P$. It enables to prove the uniqueness of Krausz coverings for some cases.

Section 6 deals with the Whitney theorem [17]. It is known that this theorem has no simple analogue for line graphs of hypergraphs. However, its versions are possible in a more general situation. One of them is related to Harary's conjecture on reconstruction of a graph from the list of its subgraphs obtained by deleting one edge. There are a number of papers [1, 3, 5–10, 15] devoted to this version where a strong isomorphism of hypergraphs ($\cong$) is mainly considered, i.e., a tougher relationship than isomorphism. The results closest to the Whitney theorem are the theorem of Berge and Rado [5] and Gardner [10]. In [5], for every integer $p \geqslant 2$, a pair of hypergraphs $\mathscr{B}_p = (V, (B_1, \dots, B_p))$ and $\mathscr{D}_p = (W, (D_1, \dots, D_p))$ is constructed in a certain way. For $p = 3$, this pair coincides with the pair excluded in the Whitney theorem: $\mathscr{B}_3 = K_{1,3}$ and $\mathscr{D} = K_3$. Now suppose that

$$H_k = (V_k, (E_1^k, \dots, E_m^k)), \quad k = 1, 2, \ m \geqslant p, \tag{1}$$

are two hypergraphs. If there exist $X_k \subseteq V_k$, $k = 1, 2$ and $I \subseteq \{1, \dots, m\}$ with $|I| = p$ such that one of the partial subhypergraphs

$$H_k(X_k, I_k) = (X_k \cap V_k, (E_i^k : i \in I))$$

is strongly isomorphic to $\mathscr{B}_p$ and the other to $\mathscr{D}_p$, then the pair $H_1, H_2$ is said to contain the $\mathscr{B}_p$–$\mathscr{D}_p$ pair.

**Theorem** (Berge and Rado [5]). (i) *Let* (1) *be two hypergraphs whose partial hypergraphs $H_1(I)$ and $H_2(I)$ are strongly isomorphic for all $I \subseteq \{1, \dots, m\}$ with $|I| = p - 1$. Suppose that the pair $H_1, H_2$ does not contain the $\mathscr{B}_p$–$\mathscr{D}_p$ pair. Then $H_1 \cong H_2$.*

(ii) *$\mathscr{B}_p \not\cong \mathscr{D}_p$ while $\mathscr{B}_p - B_k \cong \mathscr{D}_p - D_k$ for every $k = 1, \dots, p$.*

Based on the concept of a $\mathscr{B}_p$–$\mathscr{D}_p$ pair and the above theorem, Gardner obtained in close terms a criterion of strong isomorphism for hypergraphs with the same line graph [10].

In this paper, another version of the Whitney theorem is developed. The Whitney theorem may be formulated as the following uniqueness theorem: *if $G$ is a simple connected graph and $P$ is the class of simple graphs whose orders are greater than* 3, *then*

$$|\mathscr{L}^{-1}(G) \cap P| \leqslant 1. \tag{2}$$

In Section 5, condition (2) for a hypergraph $G$ is connected with acting the automorphism group Aut($G$). It enables us to prove inequality (2) for a number of classes of hypergraphs and some given properties $P$. In particular, we obtain a direct generalization of the Whitney theorem.

Some results of this paper in a less general form have been obtained earlier in [12]. Moreover, the concept of a line hypergraph is useful for study of representation of independence systems as an intersection of matroids [16].

Another generalization of line graphs can be found in [14].

## 2. Basic terminology

With minor adaptations, we adopt the terminology of Berge [2]. Recall some definitions and note differences from Berge's terminology.

A *hypergraph* is a pair $(V, \mathscr{E})$, where $V$ is a finite nonempty set (*the vertex set*) and $\mathscr{E}$ is a finite family of nonempty subsets of $V$ (*the edge family*). In contrast to the generally accepted definition, isolated vertices are permitted. This is necessary, since the line graph of any hypergraph must be a hypergraph.

The vertex set and the edge family of a hypergraph $H$ are denoted by $VH$ and $\mathscr{E}H$, respectively. The number $|VH|$ is called the *order* of $H$ and is denoted by $|H|$.

For a vertex $v \in VH$, the edge subfamily

$$(E \in \mathscr{E}H \colon v \in E) = \mathscr{E}_H(v) = \mathscr{E}(v)$$

is called the *star* of the vertex $v$. Note that $\mathscr{E}(v)$ is a family of edges but not a partial hypergraph $H(v)$ as in [4]. The family $(\mathscr{E}(v) \colon v \in VH) = \mathscr{S}H$ is called the *star family* of $H$.

The number $|\mathscr{E}(v)| = \deg v$ is the *degree* of a vertex $v \in VH$, and $|E|$ is the *degree* of an edge $E \in \mathscr{E}H$. An edge of degree 1 is called a *loop*.

Further,

$$\operatorname{rank} H = \max_{B \in \mathscr{E}H} \deg E.$$

A hypergraph without multiple edges is called a *simple hypergraph*. The edge family of a simple hypergraph $H$ can be considered as a set. In this case, we write $EH$ instead of $\mathscr{E}H$ as for simple graphs.

The *union* $G \cup H$ of simple hypergraphs $G$ and $H$ is defined as follows:

$$V(G \cup H) = VG \cup VH, \qquad E(G \cup H) = EG \cup EH.$$

A hypergraph is called an *antichain* if no edge is a subset of another edge. We say that a hypergraph is *l-linear* ($l \geqslant 1$) if no two edges have $l + 1$ common vertices. In particular, the class of 1-linear hypergraphs is exactly the well-known class of *linear* hypergraphs.

A hypergraph is called an *r-uniform* (*or uniform*) *hypergraph* if all its edges have the same degree $r$. The *complete r-uniform hypergraph* $K_n^r$ is a simple hypergraph of order $n$ whose edge set coincides with the set of $r$-subsets of the vertex set ($1 \leqslant r \leqslant n$).

A hypergraph $G$ is called a *partial hypergraph* of a hypergraph $H$ is $\mathscr{E}G \subseteq \mathscr{E}H$ and $VG = \bigcup_{E \in \mathscr{E}G} E$. If a partial hypergraph of $H$ is a connected complete uniform hypergraph, then it is called the *clique* of $H$. More precisely, if $r \geqslant 2$, then $K_n^r$ is a *clique of rank r*; $K_1^1$ is a *clique of rank* 1; a single vertex is a *clique of rank* 0. A *maximal clique* is maximal with respect to inclusion.

A finite family

$$Q = (C_j \colon j \in J) \tag{3}$$

of cliques $C_j$ is called a *clique covering* of a hypergraph $H$ if $H$ is the union of cliques from $Q$. The cliques $C_j$ are called the *components* and the family

$$\operatorname{rank} Q = (r_j\colon j \in J), \quad r_j = \operatorname{rank} C_j$$

is called the *rank* of $Q$. A component of rank 0 is called *trivial*. The covering $Q$ is called *irreducible* if no clique $C_j$ is a partial hypergraph of other clique from $Q$. The minimal number of cliques taken over all clique coverings of $H$ is denoted by $\operatorname{cc}(H)$. Evidently that simple hypergraphs only have clique coverings.

We say that two clique coverings $(C_j\colon j \in J_1)$ and $(D_j\colon j \in J_2)$ are *equal* if there exists a bijection $\alpha\colon J_1 \to J_2$ such that $C_j = D_{\alpha(j)}$ for every $j \in J_1$.

A *covering* and the *equality* of coverings for the vertex set $VH$ are defined analogously. The vertex set of a clique will be called a *clique* too if it makes no confusion.

If (3) is a clique covering of a hypergraph $H$, then the family

$$VQ = (VC_j\colon j \in J)$$

is a clique covering of the vertex set $VH$.

Let $H_1$ and $H_2$ be two hypergraphs and let

$$\mathscr{E}H_k = (E_i^k\colon i \in I_k), \quad k = 1, 2.$$

An *isomorphism* $(\alpha, \beta)\colon H_1 \to H_2$ is a pair of bijections

$$\alpha\colon VH_1 \to VH_2 \quad \text{and} \quad \beta\colon I_1 \to I_2$$

such that if $E_i^1 = \{v_j\colon 1 \leqslant j \leqslant r\}$ then

$$\alpha(E_i^1) = \{\alpha(v_j)\colon 1 \leqslant j \leqslant r\} = E_{\beta(i)}^a.$$

If there exists an isomorphism $H_1 \to H_2$, then we say that $H_1$ and $H_2$ are *isomorphic* and write $H_1 \simeq H_2$. For simple hypergraphs, their isomorphism is defined more easily. This is a bijection $\alpha\colon VH_1 \to VH_2$ such that $\alpha(X) \in EH_2$ if and only if $X \in EH_1$ for any subset $X \subseteq VH_1$.

For a hypergraph $H$, the *line graph* $L(H)$ is defined as follows:

(i) $VL(H) = \mathscr{E}H$. In accordance with the definition of a hypergraph, $VL(H)$ is a set and $\mathscr{E}H$ is a family. In this situation the above equality means that if $\mathscr{E}H = (E_i\colon 1 \leqslant i \leqslant m)$, then $VL(H) = \{E_1, \dots, E_m\}$ is an $m$-element set. In other words, multiple edges of $H$ give rise to different vertices of $L(H)$;

(ii) vertices $E_i$ and $E_j$ are adjacent in $L(H)$ if and only if $E_i \cap E_j \neq \emptyset$.

The *dual hypergraph* $H^*$ of a hypergraph $H$ without isolated vertices is the following object. The vertices of $H^*$ are exactly the edges of $H$ and the edges of $H^*$ are exactly the vertices of $H$. A vertex $E$ and an edge $v$ are incident in $H^*$ if and only if the corresponding edge $E$ and vertex $v$ are incident in $H$.

The 2-*section* $[H]_2$ without loops of a hypergraph $H$ is defined as the simple graph whose vertex set is $VH$ and two different vertices are adjacent if and only if they are adjacent in $H$.

It is obvious that $H^*$ can be realized in the form:

$$VH^* = \mathscr{E}H, \qquad \mathscr{E}H^* = (E_v\colon v \in VH), \qquad E_v = \mathscr{E}_H(v).$$

For $L(H)$ we have

$$L(H) = \bigcup_{v \in VH} F_v, \tag{4}$$

where $F_v$ is the complete graph with the vertex set $\mathscr{E}_H(v)$. Therefore,

$$[H^*]_2 = L(H). \tag{5}$$

A hypergraph $H$ is called *conformal* if the vertex set of any clique in the 2-section $[H]_2$ is contained in some edge of $H$.

A hypergraph $H$ without isolated vertices with $\mathscr{E}H = (E_i\colon i \in I)$ is said *to satisfy the Helly property* if $J \subseteq I$ and $E_i \cap E_j \neq \emptyset$ for all $i, j \in J$ implies that $\bigcap_{j \in J} E_j \neq \emptyset$.

## 3. Definition of the function 'line hypergraph'

Let $H$ be a hypergraph without isolated vertices. Compare the line graph $L(H)$ and the dual hypergraph $H^*$. According to equality (4), $L(H)$ is obtained from $H^*$ by replacing each edge $E$ by a complete graph (i.e., a clique of rank 2) with the vertex set $E$. Possible multiple edges are identified. If we begin to replace edges of $H^*$ by cliques of different ranks but not only 2, then we obtain hypergraphs 'similar' to both $L(H)$ and $H^*$. In particular, for any such a hypergraph $H'$, equality (5) holds, an idea of a multivalued function $\mathscr{L}$ whose set of values is the set of hypergraphs $H'$ naturally appears.

To put it more precisely, let $H$ be a hypergraph without isolated vertices, $VH = \{v_1, \dots, v_n\}$ be its vertex set, and $1_H = (\deg v_i\colon 1 \leqslant i \leqslant n)$ be the degree sequence of $H$. Put

$$0_H = (0_{v_i}\colon 1 \leqslant i \leqslant n), \quad \text{where } 0_{v_i} = \begin{cases} 0 & \text{if } \deg v_i = 1, \\ 2 & \text{if } \deg v_i > 1. \end{cases}$$

Furthermore, let $\mathbf{Z}_+^n$ be the lattice of integer-valued strings $x = (x_1, \dots, x_n)$, $x_i \geqslant 0$, $1 \leqslant i \leqslant n$, with the following order:

$$x \leqslant y \Leftrightarrow x_i \leqslant y_i, \quad 1 \leqslant i \leqslant n.$$

At last, let $\mathscr{D}_H = [0_H, 1_H]$ be an interval in $\mathbf{Z}_+^n$ and $\mathscr{D} = (d_1, \dots, d_n) \in \mathscr{D}_H$. For $v_i \in VH$, $F_{v_i}$ denotes the clique of rank $d_i$ with the vertex set $\mathscr{E}(v_i)$. Put

$$\mathscr{L}_{\mathscr{D}}(H) = \bigcup_{i=1}^{n} F_{v_i}.$$

The hypergraph $\mathscr{L}_{\mathscr{D}}(H)$ is called the *line hypergraph of $H$ with respect to the vector $\mathscr{D}$*.

If we write $\mathscr{D}$ in the form $\mathscr{D} = (d_v: v \in VH)$, where $d_v = d_i$ for $v = v_i$, then the previous definition takes the form

$$\mathscr{L}_D(H) = \bigcup_{v \in VH} F_v .$$

Denote by $\mathscr{H}$ the set of hypergraphs without isolated vertices and define the function $\mathscr{L}$ on $\mathscr{H}$ as follows:

$$\mathscr{L}(H) = \{\mathscr{L}_{\mathscr{D}}(H): \mathscr{D} \in \mathscr{D}_H\}, \quad H \in \mathscr{H} .$$

The function $\mathscr{L}$ is called the *line hypergraph*. Any element from the image $\mathscr{L}(H)$ is called a *line hypergraph of H*. Hypergraphs in $\mathscr{L}(H)$ differ up to isomorphism, and they all are simple.

It is evident that $\mathscr{L}_{0_H}(H) = L(H)$ and $\mathscr{L}_{1_H}(H)$ is obtained from the dual hypergraph $H^*$ as the result of identifying multiple edges. If $H$ does not contain similar vertices, i.e., $\mathscr{E}(v_i) \neq \mathscr{E}(v_j)$ for $i \neq j$, then $\mathscr{L}_{1_H}(H) = H^*$. It is also evident that $|\mathscr{L}(H)| = 1$ if and only if $1_H = (2, \ldots, 2)$, i.e., $H^*$ is a multigraph.

**Example 1.** For $H = K_2$ we have $1_H = (1,1), 0_H = (0,0), \mathscr{D}_H = (d_1, d_2): 0 \leqslant d_1 \leqslant 1\} = \{\mathscr{D}_i: 1 \leqslant i \leqslant 4\}$, where

$$\mathscr{D}_1 = 1_H, \qquad \mathscr{D}_2 = (1,0), \qquad \mathscr{D}_3 = (0,1), \qquad \mathscr{D}_4 = 0_H .$$

Hence $\mathscr{L}(H)$ takes two values:

$$\mathscr{L}_{\mathscr{D}_1}(H) = \mathscr{L}_{\mathscr{D}_2}(H) = \mathscr{L}_{\mathscr{D}_3}(H) = K_1^1 \quad \text{and} \quad \mathscr{L}_{\mathscr{D}_4}(H) = L(H) = K_1 .$$

**Example 2.** For $H = K_3$ we have

$$1_H = 0_H = (2,2,2) \quad \text{and} \quad \mathscr{L}(K_3) = L(K_3) = (K_3)^* = K_3 .$$

**Example 3.** Let $H = K_{1,3}$. Since $1_H = (3,1,1,1)$, we have $0_H = (2,0,0,0)$ and $\mathscr{D}_H = \{(d_i: 1 \leqslant i \leqslant 4): \ 2 \leqslant d_1 \leqslant 3, \ 0 \leqslant d_i \leqslant 1 \ \text{for} \ i = 2,3,4\}$, $|\mathscr{D}_H| = 16$. However, some pairs of vectors from $\mathscr{D}_H$ yield isomorphic line hypergraphs. So, the function $\mathscr{L}$ takes 8 values shown in Fig. 1. These values correspond to the following vectors:

$$\mathscr{D}_1 = (3,1,1,1), \qquad \mathscr{D}_2 = (3,1,1,0), \qquad \mathscr{D}_3 = (3,1,0,0), \qquad \mathscr{D}_4 = (3,0,0,0),$$

$$\mathscr{D}_5 = (2,1,1,1), \qquad \mathscr{D}_6 = (2,1,1,0), \qquad \mathscr{D}_7 = (2,1,0,0), \qquad \mathscr{D}_8 = (2,0,0,0),$$

$$\mathscr{D}_1 = 1_H \quad \text{and} \quad \mathscr{D}_8 = 0_H .$$

It follows from (4) that the family of cliques $(F_v: v \in VH)$ is a covering of the line hypergraph $\mathscr{L}_{\mathscr{D}}(H)$. Let us call this covering *standard* and denote it by $S(H,D)$.

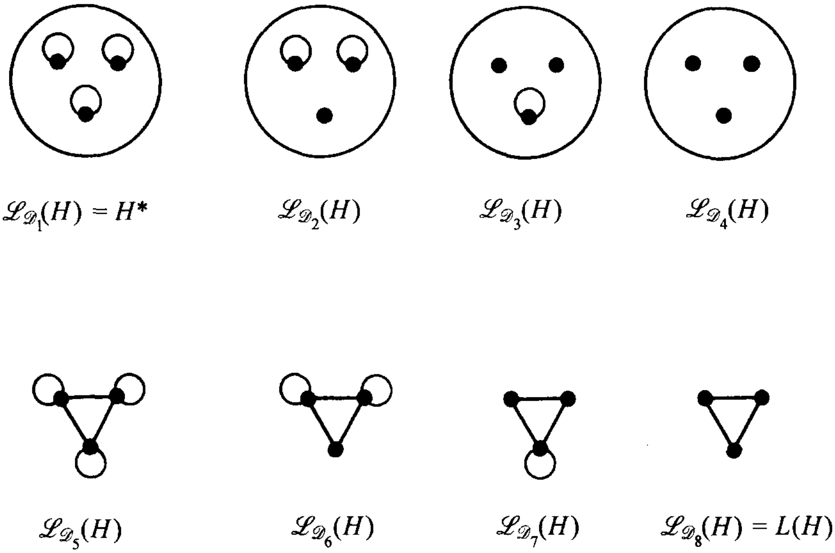


Fig. 1. Line hypergraphs of star $K_{1,3}$.

## 4. On isomorphisms of line hypergraphs

Let $H_1$ and $H_2$ be two hypergraphs without isolated vertices:

$$VH_k = V_k, \quad \mathscr{E}H_k = (E_i^k : i \in I_k), \qquad k = 1, 2.$$

Further, let $\mathcal{D}_k$ be a vector from the interval $\mathcal{D}_{H_k}$ (Section 3) and

$$G_k = \mathcal{L}_{\mathcal{D}_k}(H_k).$$

At last, let

$$\alpha : V_1 \to V_2, \qquad \beta : I_1 \to I_2, \qquad \varphi = (\alpha, \beta) : H_1 \to H_2 \tag{6}$$

be an isomorphism of hypergraphs. Isomorphism (6) is called an *isomorphism of pairs* $(H_1, \mathcal{D}_1) \to (H_2, \mathcal{D}_2)$ if

$$d_v^1 = d_{\alpha(v)}^2$$

for every vertex $v \in V_1$, where $(d_v^k : v \in V_k) = \mathcal{D}_k$.

Define a mapping $\bar{\varphi} = \bar{\beta}$ as follows:

$$\bar{\beta} : VG_1 \to VG_2, \qquad E_i^1 \to E_{\beta(i)}^2.$$

It is evident that $\bar{\varphi}$ is a bijection. We say that $\bar{\varphi}$ *is induced* by the isomorphism $\varphi$.

Let $\bar{\delta} : VG_1 \to VG_2$ be a bijection. We shall say that $\bar{\delta}$ *preserves stars* if there exists a bijection $\gamma : V_1 \to V_2$ such that

$$\bar{\delta}(\mathscr{E}_{H_1}(v)) = \mathscr{E}_{H_2}(\gamma(v)) \tag{7}$$

for every vertex $v \in V_1$.

The bijection $\bar{\delta}$ naturally acts on the star family $\mathscr{S}H_1$:

$$\bar{\delta}(\mathscr{S}H_1) = (\bar{\delta}(\mathscr{E}_{H_1}(v))\colon v \in V_1).$$

The star family $\mathscr{S}H_k$ is a covering of the vertex set $VG_k$ and by definition of the equality of coverings (Section 2), equalities (7) give

$$\bar{\delta}(\mathscr{S}H_1) = \mathscr{S}H_2.$$

Consider standard coverings

$$S(H_k, \mathscr{D}_k) = (F_v^k\colon v \in V_k)$$

(Section 3). Suppose that in addition to (7) the following conditions hold

$$d_v^1 = d_{\gamma(v)}^2, \quad v \in V_1. \tag{8}$$

Equalities (7) and (8) mean that $\bar{\delta}$ induces an isomorphism of cliques $F_v^1 \to F_{\gamma(v)}^2$, i.e., $\bar{\delta}(F_v^1) = F_{\gamma(v)}^2$, and by definition of the equality of coverings

$$\bar{\delta}(S(H_1, \mathscr{D}_1)) = S(H_2, \mathscr{D}_2)).$$

In this case, we say that $\bar{\delta}$ *preserves a standard covering.*

It is evident that for any isomorphism $\varphi$ the induced mapping $\bar{\varphi}$ preserves stars. And if $\varphi$ is an isomorphism of pairs, then $\bar{\varphi}$ preserves a standard covering.

**Lemma 1.** *For every bijection $\bar{\delta}\colon VG_1 \to VG_2$ preserving stars there exists an isomorphism of hypergraphs $H_1 \to H_2$ inducing $\bar{\delta}$. If $\bar{\delta}$ preserves a standard covering, then it is an isomorphism of line hypergraphs $G_1 \to G_2$ such that there exists an isomorphism of pairs $(H_1, \mathscr{D}_1) \to (H_2, \mathscr{D}_2)$ inducing $\bar{\delta}$.*

**Proof.** Let $\bar{\delta}\colon VG_1 \to VG_2$ be a bijection preserving stars. Then there exists a bijection $\gamma$ satisfying (7). Define the bijection $\delta\colon I_1 \to I_2$ as follows:

$$\delta(i) = j \iff \bar{\delta}(E_i^1) = E_j^2 \quad \text{for } i \in I_1, \tag{9}$$

and show that the pair $(\gamma, \delta)$ is an isomorphism of hypergraphs $H_1 \to H_2$. If

$$E_i^1 \in \mathscr{E}H_1, \qquad \bar{\delta}(E_i^1) = E_j^2, \quad \text{and} \quad E_i^1 = \{v_1, \dots, v_d\},$$

then

$$E_i^1 \in \mathscr{E}_{H_1}(v_1) \cap \cdots \cap \mathscr{E}_{H_1}(v_d).$$

It follows from (7) that

$$E_j^2 \in \mathscr{E}_{H_2}(\gamma(v_1)) \cap \cdots \cap \mathscr{E}_{H_2}(\gamma(v_d))$$

and

$$E_j^2 = \{\gamma(v_1), \dots, \gamma(v_d), \dots\}.$$

If $\gamma(v_{d+1}) \in E_j^2$, then

$$E_j^2 \in \mathscr{E}_{H_2}(\gamma(v_{d+1})), \quad E_i^1 \in \mathscr{E}_{H_1}(v_{d+1}), \qquad v_{d+1} \in E_i^1 .$$

This is impossible since $|E_i^1| = d$. Consequently,

$$E_j^2 = \{\gamma(v_1), \ldots, \gamma(v_d)\},$$

and thus

$$\gamma(E_i^1) = \{\gamma(v_1), \ldots, \gamma(v_d)\} = E_j^2 = \bar{\delta}(E_i^1) = E_{\delta(i)}^2,$$

i.e., $\psi = (\gamma, \delta)$ is an isomorphism of hypergraphs $H_1 \to H_2$. It follows from (9) that $\psi$ induces the bijection $\bar{\delta}$. By definition, $\psi$ is an isomorphism of pairs $(H_1, \mathscr{D}_1) \to (H_2, \mathscr{D}_2)$ if (8) holds.

It remains to prove that $\bar{\delta}: G_1 \to G_2$ is an isomorphism of hypergraphs. Suppose that $\emptyset \neq X \subseteq VG_1$. If $X \in EG_1$ then, by (4), $X \subseteq \mathscr{E}_{H_1}(v)$ for some vertex $v \in V_1$ and $|X| = d_v^1$. Taking (7) and (8) into account, we have

$$\bar{\delta}(X) \subseteq \mathscr{E}_{H_2}(\gamma(v)), \qquad |\bar{\delta}(X)| = d_v^1 = d_{\gamma(v)}^2 .$$

Thus, $\bar{\delta}(X) \in EG_2$.

Using the inverse bijection $\bar{\delta}^{-1}$, we get analogously

$$\bar{\delta}(X) \in EG_2 \;\Rightarrow\; X = \bar{\delta}^{-1}\bar{\delta}(X) \in EG_1 .$$

Thus,

$$\bar{\delta}(X) \in EG_2 \;\Leftrightarrow\; X \in EG_1 .$$

Consequently, $\bar{\delta}$ is an isomorphism of hypergraphs $G_1$ and $G_2$. □

The next two corollaries are obvious.

**Corollary 1.** (i) *$H_1 \simeq H_2$ if and only if there exists a bijection $VG_1 \to VG_2$ preserving stars;*

(ii) *$(H_1, \mathscr{D}_1) \simeq (H_2, \mathscr{D}_2)$ if and only if there exists a bijection $VG_1 \to VG_2$ preserving standard covering (this bijection is an isomorphism of line hypergraphs $G_1 \to G_2$).*

Following the terminology of Lovasz [13], we say that an isomorphism of line hypergraphs $G_1 \to G_2$ is *trivial* if there exists an isomorphism $H_1 \to H_2$ inducing it. In the next corollary, the statement from [13, p. 506] on trivial isomorphisms of line graphs is transferred onto line hypergraphs.

**Corollary 2.** (i) *If $\varphi$ is an isomorphism of pairs $(H_1, \mathscr{D}_1) \to (H_2, \mathscr{D}_2)$, then $\bar{\varphi}$ is a trivial isomorphism of line hypergraphs;*

(ii) *An isomorphism $\bar{\delta}: G_1 \to G_2$ is trivial if and only if it preserves stars. If $\bar{\delta}$ is trivial, then the isomorphism $H_1 \to H_2$ inducing $\bar{\delta}$ can be chosen among isomorphisms of pairs $(H_1, \mathscr{D}_1) \to (H_2, \mathscr{D}_2)$.*

## 5. Inverse image $\mathscr{L}^{-1}(G)$

It was pointed out above that all values of the function $\mathscr{L}(H)$ are simple hypergraphs for any hypergraph $H$. Let $G$ be an arbitrary simple hypergraph. The set of hypergraphs $H$ such that $\mathscr{L}_{\mathscr{D}}(H) \simeq G$ for some $\mathscr{D} \in \mathscr{D}_H$ is called the *inverse image* $\mathscr{L}^{-1}(G)$.

The aim of further considerations is investigations of $\mathscr{L}^{-1}(G)$ for given $G$. This develops Berge's idea [2, p. 400]. For a simple graph $G$, the inverse image $\mathscr{L}^{-1}(G)$ is described in [2] in terms of clique coverings of $G$. That results is generalized here.

Let $G$ be a simple hypergraph, $\mathscr{A}_G$ be the set of triads $(H, \mathscr{D}, \gamma)$, where $H \in \mathscr{L}^{-1}(G)$, $\mathscr{D} \in \mathscr{D}_H$, $\mathscr{L}_{\mathscr{D}}(H) \simeq G$ and $\gamma : \mathscr{L}_{\mathscr{D}}(H) \to G$ is an isomorphism. Let $\mathscr{B}_G$ be the set of clique coverings of $G$. Obviously that $\mathscr{B}_G \neq \emptyset$.

The concept of a canonical hypergraph plays an important role here. Let

$$Q = (C_i \colon 1 \leqslant i \leqslant l) \in \mathscr{B}_G .$$

Define the hypergraph $F_Q$ as follows:

$$VF_Q = VG, \qquad \mathscr{E}F_Q = (E_i \colon 1 \leqslant i \leqslant l), \qquad E_i = VC_i . \tag{10}$$

It is clear that $F_Q$ does not contain isolated vertices. Hence there exists the dual hypergraph $(F_Q)^*$. The hypergraph $(F_Q)^*$ is called *canonical* (*with respect to* $Q$) and is denoted by $C(Q)$.

For graphs, the construction of a canonical hypergraph was used in [2]. The concept of a canonical hypergraph was introduced by Gardner [10] for the particular case when $Q$ is a covering of edges.

**Lemma 2.** rank $Q \in \mathscr{D}_{C(Q)}$. *Consequently, there exists a line hypergraph* $\mathscr{L}_{\mathscr{D}}(C(Q))$ *with* $\mathscr{D} = \operatorname{rank} Q$.

**Proof.** By definition,

$$VC(Q) = \{E_i \colon 1 \leqslant i \leqslant l\}, \qquad E_i = VC_i . \tag{11}$$

If $VG = \{v_j \colon 1 \leqslant j \leqslant n\}$, then

$$\mathscr{E}C(Q) = \mathscr{S}F_Q = (V_j \colon 1 \leqslant j \leqslant n\}, \qquad V_j = \mathscr{E}_{F_Q}(v_j) .$$

For $E_i \in VC(Q)$, we have

$$\mathscr{E}_{C(Q)}(E_i) = (V_j \colon E_i \in V_j) = (V_j \colon v_j \in E_i) . \tag{12}$$

Consequently, $\deg_{C(Q)} E_i = |E_i| = |C_i|$. Therefore, $\operatorname{rank} C_i \leqslant |E_i|$ and the lemma is proved. $\square$

Put

$$\mathscr{D}_Q = \operatorname{rank} Q = (d_i\colon 1 \leqslant i \leqslant l).$$

The pair $(C(Q), \mathscr{D}_Q)$ is called the *canonical pair* $\mathrm{CP}(Q)$.

Define the mapping

$$\gamma_Q\colon V\mathscr{L}_{\mathscr{D}_Q}(C(Q)) \to VG, \qquad V_j \to v_j,\ 1 \leqslant j \leqslant n.$$

The triad $(C(Q), \mathscr{D}_Q, \gamma_Q)$ is called the *canonical triad* $\mathrm{CT}(Q)$.

**Lemma 3.** *The mapping $\gamma_Q$ is an isomorphism of hypergraphs $\mathscr{L}_{\mathscr{D}_Q}(C(Q)) \to G$, and $CT(Q) \in \mathscr{A}_G$.*

**Proof.** Denote $C(Q) = H'$ and consider an arbitrary subset

$$E = \{V_{i_1}, \ldots, V_{i_d}\} \subseteq V\mathscr{L}_{\mathscr{D}_Q}(H').$$

Evidently, $E \in E\mathscr{L}_{\mathscr{D}_Q}(H')$ if and only if there is a vertex $E_i \in VH'$ such that

$$E \subseteq \mathscr{E}_{H'}(E_i) \quad \text{and} \quad d = d_i. \tag{13}$$

According to (12), condition (13) is equivalent to the following:

$$\{v_{i_1}, \ldots, v_{i_d}\} \subseteq E_i, \qquad d = d_i = \operatorname{rank} C_i,$$

i.e., $\gamma_Q(E) = \{v_{i_1}, \ldots, v_{i_d}\} \in EG$. Thus, $\gamma_Q\colon \mathscr{L}_{\mathscr{D}_Q}(H') \to G$ is an isomorphism □

Define the mapping

$$\psi\colon \mathscr{A}_G \to \mathscr{B}_G, \qquad (H, \mathscr{D}, \gamma) \to \gamma(S(H, \mathscr{D})),$$

where $S(H, \mathscr{D})$ is a standard covering.

**Theorem 1.** *Under the previous notation the following three assertions hold:*

(i) *The mapping $\psi$ is a surjection. If $Q$ is an arbitrary clique covering of $G$, then*

$$\psi(\mathrm{CT}(Q)) = Q.$$

(ii) Let

$$(H_k, \mathscr{D}_k, \gamma_k) \in \mathscr{A}_G, \quad k = 1, 2. \tag{14}$$

*Put $Q_k = \psi(H_k, \mathscr{D}_k, \gamma_k)$ and let $VQ_k$ be the clique covering of the vertex set $VG$ corresponding to the clique covering $Q_k$. Then $H_1 \cong H_2$ if and only if there exists a bijection $\alpha\colon VG \to VG$ such that*

$$\alpha(VQ_1) = VQ_2. \tag{15}$$

(iii) $(H_1, \mathscr{D}_1) \simeq (H_2, \mathscr{D}_2)$ *for triads* (14) *if and only if there exists an automorphism* $\alpha \in \mathrm{Aut}(G)$ *such that* $\alpha(Q_1) = Q_2$.

**Proof.** (i) By Lemma 3, $\mathrm{CT}(Q) \in \mathscr{A}_G$. Find $\psi(\mathrm{CT}(Q))$. Since $\mathrm{CT}(Q) = (C(Q), \mathscr{D}_Q, \gamma_Q)$, it follows that $\psi(\mathrm{CT}(Q))$ coincides with the image of the standard covering $S(C(Q), \mathscr{D}_Q)$ under acting the isomorphism $\gamma_Q$. Let $Q = (C_i\colon 1 \leqslant i \leqslant l)$. The above standard covering is the family of cliques $F_v$ of the line hypergraph $\mathscr{L}_{\mathscr{D}_Q}(C(Q))$, where $v$ is taken over all $VC(Q)$ in accordance with (11). The vertex set of the clique $F_{E_i}$ is the star $\mathscr{E}_{C(Q)}(E_i)$ and $\operatorname{rank} F_{E_i} = d_i$, where $(d_i\colon 1 \leqslant i \leqslant l) = \operatorname{rank} Q$. By (12),

$$VF_{E_i} = (V_j\colon v_i \in E_j),$$

where $\{v_j\colon 1 \leqslant j \leqslant n\} = VG$. Consequently, $\gamma_Q(F_{E_i})$ is a clique of rank $d_i$ with the vertex set $\{v_j\colon v_j \in E_i\} = E_i = VC_i$. Thus, $\psi(\mathrm{CT}(Q)) = Q$.

(ii) This follows immediately from Corollary 1. Indeed, by this corollary, $H_1 \simeq H_2$ if and only if there exists a bijection $\bar{\delta}\colon V\mathscr{L}_{\mathscr{D}_1}(H_1) \to V\mathscr{L}_{\mathscr{D}_2}(H_2)$ preserving stars. Now, if there exists $\bar{\delta}$, then put $\alpha = \gamma_2 \bar{\delta} \gamma_1^{-1}$. Obviously, $\alpha$ is a bijection on the set $VG$. Further, $VQ_k = \gamma_k(\mathscr{S}H_k)$, where $\mathscr{S}H_k$ is the star family of $H_k$. Hence

$$\alpha(VQ_1) = \gamma_2 \bar{\delta} \gamma_1^{-1} \gamma_1(\mathscr{S}H_1) = \gamma_2 \bar{\delta}(\mathscr{S}H_1) = \gamma_2(\mathscr{S}H_2) = VQ_2.$$

Conversely, if $\alpha\colon VG \to VG$ is a bijection satisfying (15), then put $\bar{\delta} = \gamma_2^{-1} \alpha \gamma_1$. We have

$$\bar{\delta}\colon V\mathscr{L}_{\mathscr{D}_1}(H_1) \to V\mathscr{L}_{\mathscr{D}_2}(H_2), \qquad \bar{\delta}(\mathscr{S}H_1) = \gamma_2^{-1} \alpha \gamma_1 \gamma_1^{-1}(VQ_1) = \mathscr{S}H_2.$$

(iii) This case is handled in the same way. The proof of Theorem 1 is complete. □

**Corollary 3.** *If $\psi(H, \mathscr{D}, \gamma) = Q$, then $(H, \mathscr{D}) \simeq \mathrm{CP}(Q)$.*

The next theorem follows directly form Theorem 1.

**Theorem 2.** (i) *For any simple hypergraph $G$, the inverse image $\mathscr{L}^{-1}(G)$ coincides with the set of canonical hypergraphs $C(Q)$, where $Q$ is taken over all clique coverings of $G$.*

(ii) *$C(Q_1) \simeq C(Q_2)$ if and only if there exists a Bijection $\bar{\delta}\colon VG \to VG$ such that*

$$\bar{\delta}(VQ_1) = VQ_2, \tag{16}$$

*where $VQ_h$ is a clique covering of $VG$ corresponding to $Q_h$.*

(iii) *$\mathrm{CP}(Q_1) \simeq \mathrm{CP}(Q_2)$ if and only if there exists an automorphism $\bar{\delta} \in \mathrm{Aut}(G)$ satisfying* (16). *The automorphism $\bar{\delta}$ satisfying* (16) *is trivial.*

(iv) *If a hypergraph $G$ is uniform, then $C(Q_1) \simeq C(Q_2)$ if and only if $\mathrm{CP}(Q_1) \simeq \mathrm{CP}(Q_2)$.*

## 6. Krausz type characterizations and coverings

The following characterization of line graphs is well known.

**Krausz theorem** [11]. *A graph G is a line graph of some simple graph if and only if there exists a clique covering of G satisfying the next two conditions*:

(i) *every vertex of G belongs to exactly two components of the covering*;

(ii) *any two components of the covering have at most one vertex in common.*

For line hypergraphs, the Krausz theorem can be generalized in the following way. Let $P$ be a hypergraph-theoretic property. We can understand $P$ as a class of hypergraphs closed with respect to isomorphism. Without loss of generality, let any hypergraph from $P$ have no isolated vertices. Put

$$P^* = \{H^* : H \in P\}, \qquad \mathscr{L}(P) = \bigcup_{H \in P} \mathscr{L}(H).$$

Furthermore, let $G$ be a simple hypergraph, $Q$ be a clique covering of $G$ and $F_Q$ be the hypergraph defined by (10). If $F_Q \in P$, then we say that $Q$ *has the property* $P$ or $Q \in P$.

**Theorem 3.** *For a hypergraph G, the following assertions are equivalent*:

(i) $G \in \mathscr{L}(P)$;

(ii) *there exists a clique covering of G having property* $P^*$.

**Proof.** From the definition of canonical hypergraph $C(Q)$ it follows that $C(Q) \in P$ if and only if $Q \in P^*$. The result now follows from Theorem 2(i). □

Many known characterizations of line graphs of hypergraphs with a given property are direct corollaries of Theorem 3 and are transferred onto line hypergraphs. Let us give some examples.

Let $P_r$ be the class of $r$-uniform hypergraphs and $P^l$ be the class of $l$-linear hypergraphs. Then $(P_r)^*$ is the class of hypergraphs whose vertex degrees are equal to $r_1$ and $(P^l)^*$ is the class of hypergraphs for which any $l+1$ edges have at most one vertex in common. In particular, $(P^l)^* = P^l$.

Let $Q$ be a clique covering of a hypergraph $G$. By definition, $Q$ is an *r-covering* if every vertex of $G$ belongs to at most $r$ components of $Q$; $Q$ is a *strict r-covering* if every vertex of $G$ belongs to exactly $r$ components of $Q$; $Q$ is an *l-linear covering* if any $l+1$ components of $Q$ have at most one vertex in common. An 1-linear covering is called a *linear covering.*

The following Corollaries 4–6 are true for any hypergraph $G$. For the case when $G$ is a graph, these corollaries are proved in [4].

**Corollary 4.** $\mathscr{L}^{-1}(G) \cap P_r \neq \emptyset$ *if and only if there exists a strict r-covering of G.*

**Corollary 5.** $\mathscr{L}^{-1}(G) \cap P^l \neq \emptyset$ *if and only if there exists an l-linear covering of G.*

Let $P_r^l$ denote the class of $l$-linear $r$-uniform hypergraphs. In particular, $P_r^{r-1}$ is the class of simple $r$-uniform hypergraphs.

**Corollary 6.** *$\mathscr{L}^{-1}(G) \cap P_r^l \neq \emptyset$ if and only if there exists an $l$-linear strict $r$-covering of $G$.*

Obviously that the word 'strict' in the statements of Corollaries 4 and 6 can be omitted.

If $l = 1$ and $r = 2$, then Corollary 6 becomes the Krausz theorem. Hence Theorem 3 and all its specifications as well as the corresponding clique coverings are called *Krausz type characterizations* and *Krausz P-coverings.*

**Corollary 7.** *Any linear hypergraph, in particular any simple graph, is a line hypergraph of some linear hypergraph.*

For graphs, Corollary 7 is proved in [4].

Furthermore, let $P = P_r$, $P^l$ or $P_r^l$, let $G \in \mathscr{L}(P)$, and let $C$ be a maximal clique of $G$. The clique $C$ is called *P-large* if

- $\operatorname{rank} C > r$ for $P = P_r$,
- $\operatorname{rank} C > l + 1$ for $P = P^l$,
- $\operatorname{rank} C > 2$ or $|C| > r^2 - r + 1$ for $P = P_r^l$.

If $\operatorname{rank} C \leqslant 1$, then $C$ is called *P-large* as well.

**Theorem 4.** *Each P-large clique of G is a component of every P-Krausz covering of G. Here the word 'strict' is omitted in the definitions of P-Krausz coverings.*

**Proof.** Let $Q$ be a $P$-Krausz covering of $G$, $A$ be a clique of $G$ such that it does not belong to any component of $Q$, $E_1 \in EA$ and $s = \operatorname{rank} A$. Construct inductively the sequences of edges $(E_i\colon 1 \leqslant i \leqslant s)$ and components $(C_i\colon 1 \leqslant i \leqslant s)$ in the following way:

$$E_1 = \{a_1, \ldots, a_s\} \in EC_1,$$

$$E_2 = \{b_1, a_2, \ldots, a_s\} \in EC_2,$$

$$\vdots$$

$$E_s = \{b_1, \ldots, b_{s-1}, a_s\} \in EC_s,$$

where $b_i \in VA \setminus VC_i$. It is obvious that all $s$ cliques $C_i$ are pairwise distinct.

If $P = P_r$, then $Q$ is an $r$-covering. Since $a_s \in VC_i$ $(1 \leqslant i \leqslant s)$, it follows that $s \leqslant r$.

If $P = P^l$, then $Q$ is an $l$-linear covering. But $\{a_{s-1}, a_s\} \subseteq VC_i$ for $1 \leqslant i \leqslant s - 1$. Hence $s - 1 \leqslant l$ or $s \leqslant l + 1$.

It remains to consider the case $P = P_r^1$ and $s = 2$. In this case, $Q$ is a linear 2-covering. Suppose that $a \in VA$ and $(D_1, \ldots, D_t)$ is the list of all components of $Q$ such that $C_i$ contains the vertex $a$ for $1 \leqslant i \leqslant t$. Then $t \leqslant r$. If

$$B_i = VA \cap VD_i = \{a, b_{i1}, \ldots, b_{im_i}\}, \quad 1 \leqslant i \leqslant t,$$

then $VA$ is divided into pairwise disjoint subsets $\{a\}$ and $B_i \backslash \{a\}$ ($1 \leqslant i \leqslant t$). It $b \in B_i$ and $c$ is taken over all $B_j$ with $j \neq i$, then all edges $bc$ belong to different components of $Q$ and $bc \notin EB_i$. Since $Q$ is an $r$-covering, it follows that $|B_j| \leqslant r - 1$ and hence $|A| \leqslant r(r-1)+1$.

Thus, the clique $A$ is not $P$-large for any of the above properties $P$. This completes the proof. □

**Corollary 8.** *If $G \in \mathscr{L}(P)$ and each maximal clique of $G$ is $P$-large, then the set of maximal cliques is a unique irreducible $P$-covering of $G$.*

**Corollary 9.** *Suppose that each maximal clique of $G$ is $P$-large. Then $G \in \mathscr{L}(P)$ if and only if the following condition holds:*

(i) *For $P = P_r$, every $r+1$ maximal cliques of $G$ have no vertex in common;*

(ii) *For $P = P^l$, every $l+1$ maximal cliques of $G$ have at most one vertex in common;*

(iii) *For $P = P_r^l$, every $r+1$ maximal cliques of $G$ have no vertex in common and every $l+1$ maximal cliques have at most one vertex in common.*

Denote by $P_{\mathrm{Hl}}$ and $P_{\mathrm{cn}}$ the classes of hypergraphs satisfying the Helly property and conformal hypergraphs, respectively.

**Corollary 10.** *Suppose that $P = P_{\mathrm{Hl}} \cap P^l$ and $G \in \mathscr{L}(P)$. Then the set of maximal cliques of $G$ is a unique irreducible $P$-covering (see, for instance,* [2]*).*

**Proof.** It is known that

$$(P_{\mathrm{Hl}})^* = P_{\mathrm{cn}}.$$

Therefore, by Theorem 3, there exists a clique covering $Q$ of $G$ such that $Q \in P_{\mathrm{cn}} \cap P^l$. Without loss of generality we can assume that $Q$ is irreducible. Let $C$ be a maximal clique of $G$. By Theorem 4, $C$ is a component of $Q$ if $\operatorname{rank} C \neq 2$. If $\operatorname{rank} C = 2$, then $C$ is a clique of the 2-section $[G]_2$. Therefore there exists a component $C' \in Q$ such that $VC \subseteq VC'$. If $E \in EC$, then $E \in EC''$ for some component $C'' \in Q$. Since the covering $Q$ is linear, we have

$$C'' = C', \qquad \operatorname{rank} C' = 2, \qquad C = C', \quad C \in Q.$$

Thus $Q$ coincides with the set of maximal cliques, since $Q$ is irreducible. The corollary is proved. □

## 7. Walking around the Whitney theorem

The Whitney theorem has no simple analogue for line hypergraphs in contrast to the Berge theorem (Section 4) and the Krausz theorem (Section 5). For the function 'line hypergraph', it is natural to consider the version of the Whitney theorem in the form of a list of double properties $(P, P')$ for which the following implication holds:

$$G \in P' \Rightarrow |\mathscr{L}^{-1}(G) \cap P| \leqslant 1. \tag{17}$$

If $P$ is the class of connected graphs of order not equal to 3 and $P'$ is the class of all graphs, then the classical Whitney theorem is a uniqueness theorem of such a form.

The next corollary follows directly from Theorem 2.

**Corollary 11.** *If $P$ is the class of uniform hypergraphs, then $|\mathscr{L}^{-1}(G) \cap P| \leqslant 1$ if and only if acting* $(\mathrm{Aut}(G), P^* \cap \mathscr{B}_G)$ *is transitive, where $\mathscr{B}_G$ is the set of clique coverings of $G$.*

Properties of $P$-coverings from Section 5 enables us to indicate a number of pairs $(P, P')$ such that implication (17) holds.

Corollaries 8 and 10 immediately imply

**Corollary 12.** *If $P''$ is the class of hypergraphs dual to antichains and $(P, P')$ is one of the pairs* (i)–(iv), *then implication* (17) *holds.*

(i) *$P = P_r$, $P'$ is the class of hypergraphs from $P''$ whose edge degrees are more than $r$;*

(ii) *$P = P^l$, $P'$ is the class of hypergraphs from $P''$ whose edge degrees are more than $l + 1$;*

(iii) *$P = P_r^l$, $P^l$ is the class of hypergraphs from $P''$ such that orders of its maximal cliques of rank 2 are more than $r(r-1)+1$;*

(iv) *$P = P^l \cap P_{Hl}$, $P' = P''$.*

Obviously that the condition $P' \subseteq P''$ is quite natural.

The classical Whitney theorem states that (17) is true if $P = P_0$ is the class of connected simple graphs of order not equal to 3 and $P'$ is the class of all simple graphs.

**Theorem 5.** *For any hypergraph $G$, implication* (17) *is true if $P = P_0$.*

To prove this theorem we need two lemmas.

**Lemma 4.** *Each clique $C$ with* $\operatorname{rank} C \neq 0$ *of a connected hypergraph $G \in \mathscr{L}(P_2^1)$ is a component of at most one linear strict 2-covering of $G$.*

**Proof.** Let $Q$ be a linear strict 2-covering of $G$, $Q = (C_i\colon i = 1, \ldots, k, \ldots)$ and $C = C_1$. Further, let $P = (C_i\colon 1 \leqslant i \leqslant k)$ be some subfamily of $Q$. Let $V$ and $E$ be the sets of vertices and edges of $G$, respectively, lying in cliques from $P$. Put $E_1 = EG \setminus E$. If $E_1 = \emptyset$, then $Q$ is obtained from $P$ by addition of trivial components, the component with a vertex $v$ being added only if $v$ belongs to exactly one component of $Q$. Thus, we can construct $Q$.

Suppose that $E_1 \neq \emptyset$. Since $G$ is connected, there exist a vertex $v \in V$ and an edge $e \in E_1$ such that $v \in e$. Denote by $E_2$ the set of edges from $E_1$ which contain the vertex $v$, and by $W$ the union of edges from $E_2$. Since the vertex $v$ can belong to at most two components of $Q$, we see that the degrees of all edges from $E_2$ are equal to $\deg e = r$. Moreover, if $H$ is a clique of rank $r$ with the vertex set $W$, then $H$ is a component of $Q$. Add this component to $P$. Repeating this construction, we obtain the covering $Q$. This yields the lemma. □

The graph $W_4$ below is a wheel shown in Fig. 3.

**Lemma 5.** *If $G$ is a connected hypergraph from $\mathscr{L}(P_2^1)$, then one of the following assertions holds*:

(i) *There exists a unique linear strict 2-covering of $G$*;

(ii) *$G$ is a graph isomorphic to $K_4 - e$, $\overline{3K_2}$, or $W_4$; there exist exactly two linear strict 2-coverings of $G$ which are interchanged by an automorphism of $G$*;

(iii) *$G \simeq K_3$ or $K_1^1$.*

**Proof.** Let $G \not\simeq K_3$ and $K_1^1$. By Theorem 2, there exists a linear strict 2-covering $Q$ of $G$. Suppose first that there is a maximal clique $A$ in $G$ such that $\operatorname{rank} A \neq 2$ or $|A| > 3$, i.e., $A$ is a $P_2^1$-large clique. By Theorem 4, the clique $A$ is a component of any linear 2-covering of $G$. In particular, $A$ is a component of $Q$. Taking Lemma 4 into account, we obtain that the covering $Q$ is a unique linear strict 2-covering of $G$.

Thus, it remains to consider the case where each maximal clique of $G$ has rank 2 and order 3, i.e., the graph $G$ is a union of triangles and does not contain $K_4$. Since $Q$ is a strict 2-covering of $G$, we have $\Delta = \Delta(G) = 3$ or 4.

Suppose that $\Delta = 3$, $a \in VG$, and $\deg a = 3$. Then the induced subgraph $F = G(N(a) \cup \{a\})$ coincides with $K_4 - e$ in Fig. 2. An arbitrary linear strict 2-covering $Q$ of the graph $G$ contains one of two triangles $T_1 = (a,b,c)$ or $T_2 = (a,b,d)$. Hence, by Lemma 4, there are at most two such coverings: $T_1 \in Q_1$ and $T_2 \in Q_2$. If $|G| = 4$, then there are exactly two such coverings:

$$Q_1 = \{T_1, da, db, \{c\}\} \quad \text{and} \quad Q_2 = \{T_2, ca, cb, \{d\}\}.$$

The transposition $(c,d)$ is an automorphism of $G$ interchanging these coverings. If $|G| > 4$, then there is the fifth vertex $f$. Without loss of generality assume that $f$ is adjacent to $c$. The vertex $f$ is not adjacent to $a$ or $b$, since $\deg a = \deg b = \Delta$. Therefore, $T_1 \in Q$ and $Q = Q_1$.

Now suppose that $\Delta = 4$, $a \in VG$, and $\deg a = 4$. Then there are three possibilities for $F$ shown in Fig. 3. For the graphs $G_1$ and $G_2$ both triangles $(a,b,c)$ and $(a,d,f)$ are components of any linear strict 2-covering. By Lemma 4, such a covering is unique. For the graph $G_3$, there are two possibilities: $(a,b,c) \in Q_1$ and $(a,b,d) \in Q_2$. If $|G| = 5$, then there exist exactly two such coverings:

$$Q_1 = \{(a,b,c),(a,d,f),bd,cf\} \quad \text{and} \quad Q_2 = \{(a,b,d),(a,c,f),bc,df\}.$$

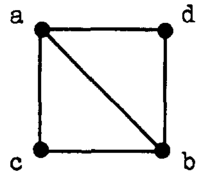


Fig. 2. Graph $K_4 - c$.

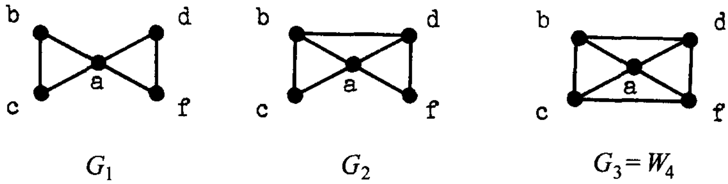


Fig. 3. Three possibilities for the graph $F$.

Moreover, $t = (c,d) \in \mathrm{Aut}(G)$ and $t(Q_1) = Q_2$. Now suppose that there is a vertex $g$ adjacent to $b$. It is evident that $g$ must be adjacent to $c$ or $d$. Assume that $g$ is adjacent to $d$. If $g$ is not adjacent to $c$, then the triangle $(b,c,a)$ is a component of any linear strict 2-covering and the result follows by Lemma 4. If $g$ is adjacent to $c$ and is not adjacent to $f$, then the same holds for $(d,f,a)$. Thus, it remains to consider only the case when $g$ is adjacent to $b,c,d$, and $f$. Since $G$ is connected and $\Delta = 4$, we have $G \cong \overline{3K_2}$. This completes the proof of Lemma 5. □

Theorem 5 follows from Corollary 11 and Lemma 5.

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